Research Project Proposal: Compact and Adaptive Basis for Shape Correspondence via Functional Maps

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1. Introduction to the problem

In the field of computer graphics, non-rigid alignment of 3D shapes plays an important role in many geometry processes. It is used, for instance, for texture transfering in digital animation and 3D objects retrieval. Other important applications are in the medical field: it can be used as a support tool for automatic measurements on digital scans or as first step in automated diagnosis. Non-rigid matching consists, basically, in finding correspondences between 3D objects related by deformations. It is a particularly challenging problem since the space of possible correspondences is huge.

A breakthrough in the field was represented by [17], which proposed to put in correspondence functions defined on the shapes rather then points. Once the bases of the functional spaces are defined, the mapping can be compactly represented as a change-of-basis matrix, as depicted in figure 1. This matrix is then determined by solving an optimization problem, namely finding the mapping that best aligns some functions defined on the shapes: descriptors (probe functions that characterize the shape), landmarks and segments. The great advantage of this approach is that such constraints are linear and the size of the mapping is independent of the size of the shape representation, making it efficiently solvable. This approach, which is referred to as functional maps, immediately reached state-of-the-art results in non-rigid alignment and was further extended by many other works, among which [16, 18, 19, 6, 13].

A key element of this framework is represented by the choice of the basis for the functional space, which needs to be:

- *compact*: it should approximate well functions even with a small number of elements
- *stable*: its span should be invariant with respect to deformation and pose change

In [17] and almost all subsequent works, the choice has been to use the first k eigenfunctions of the Laplace-Beltrami operator, which are the manifold equivalent of the Fourier basis [8, 22]. Since they are ordered in increasing frequency, this representation corresponds to a low-pass filter approximation. While this has been proven to be optimal for smooth functions [1], it is not always well suited for representation and transfer of functions with a high-level of detail [3]. By increasing the number of elements, thus covering higher frequencies, the basis becomes unstable under moderate deformations and it is harder to align. The problem of finding a good balance in the trade-off between compactness and quality of the approximation is thus still open.

2. Main related works

Many works have been proposed to try to replace or extend the basis given by the first Laplace-Beltrami eigenfunctions. Some of them tackle a specific class of functions, like [11], which defines a frame to represent step functions on manifolds. [12] proposes a basis specifically for the task of mesh transfer between shapes, by including information about the cartesian coordinates of the points. Involving extrinsic information, this approach is not deformation-invariant and thus only applicable when shapes are already almost aligned in pose. [14] proposes a localized set of functions that naturally *extend* the LB eigenbasis, but requires a region to be specified.

(a) source shape (b) target shape (c) correspondence matrix

Figure 1: An example of mapping between shapes, rendered as color correspondences, and the related matrix. Source: [17]

In [10] a learning-based approach is adopted to define a basis (together with optimized descriptors), but this requires a set of pairs of shapes with known point-to-point correspondence.

Recently, the traditional LB eigenbasis has been extended by considering the pointwise product of its elements, basing on the theoretical consideration that a functional map corresponds to a point-to-point map if and only if it preserves pointwise product of functions. In [15] the map between eigenproducts is completely derived from the alignment of eigenfunctions alone, thus being very susceptible to noise in the map. In [9] the extended base (made by eigenfunctions and their pointwise products) is orthogonalized and a new, more stable, method for recovering the mapping is proposed, but requiring a conversion to and from a point-to-point map. What is interesting about these last works, though, is the amount of high-frequency information present in eigenproducts and the fact that they could be, theoretically, aligned just by using a (very accurate) map between eigenfunctions.

3. Research plan

The goal of the research is to find a compact, yet powerful basis for the space of real-valued functions on bi-dimensional discrete manifolds. With "powerful" we mean its ability to represent and transfer functions accurately, even when they have a significant content in high frequencies. The general idea is to apply signal processing techniques, in particular (sparse) representation on overcomplete dictionaries, to tackle this problem.

This kind of approach is widely and satisfactorily used in other fields, such as image processing, where high-dimensional data need to be represented compactly. Despite this fact, it has not been much explored in the field of 3D geometry processing, in particular not for function representation. Some works have actually appeared, but they differ significantly from our approach. [7] presents some dictionary-based methods for 3D modeling, but not dealing with function representation over the shapes. In [23] an overcomplete dictionary is learned from probe functions on a set of shapes, but with the purpose of building a compact global descriptor for object retrieval. More similar to ours is [21], where a new functional basis is learned from a collection of shapes using descriptors defined on them. While the purpose is similar, they use neural networks to achieve it and the dictionary is completely built from scratch.

Our approach, instead, is based on two fundamental steps:

- **Creation of an overcomplete dictionary**, with the aim of covering a large and rich functional space. The first candidate are the pointwise products of Laplace-Beltrami eigenfunctions, due to their promising behaviour in high frequency [15, 9] and their tight connection with the current standard. Another option we would like to explore is the use of descriptors themselves, as done in [21].
- **Dimensionality reduction of the overcomplete dictionary**, basing on the assumption that not all the spanned space is equally informative. First, we would like to test PCA here, due to its simplicity. Secondly, the double sparsity framework [20], which builds a new, compact dictionary on top of a larger one. The new

atoms are sparse linear combinations of the elements of the underlying (overcomplete) dictionary. Sparsity is meant as a form of regularization and we will analyze its impact in the compactness trade-off.

Note that our approach is *adaptive*: the reduction will be performed designing a suitable pool of representative functions, that will provide the criterion to actuate the selection.

Evaluation

The research will be focused on *designing* and *implementing* a procedure to build a basis with the characteristics detailed above, which will be evaluated *experimentally* on existing shape datasets, like FAUST [4], TOSCA [5] and SCAPE [2]. The most natural setting for testing such basis is function transfer and shape correspondence through functional maps. To this purpose we will use two metrics:

- the *geodesic error* for shape matching, both in terms of average error and of percentage of point correspondences as a function of the maximum geodesic error allowed
- the *normalized approximation error* for function transfer

Both the datasets and the metrics are standards in the field and will provide an immediate comparison with current state-of-the-art methods.

Organization

The research is decomposed in the following tasks (organized temporally in figure 2):

- **Study of the state of the art**: study the literature on the functional map framework, with particular focus on the basis aspect, and on the main techniques for sparse representation of signals.
- **Familiarization**: experiment with the code of existing methods to get familiarity with them, with their technical aspects and with the tools.
- **Design**: define the method to obtain the new basis by properly selecting the initial dictionary, the reduction technique and the selection function set.
- **Implementation**: implement the code to realize the designed method, with attention in choosing efficient implementation of algorithms.
- **Testing**: test the behavior of the basis on shape datasets to assess its performance and compare to current methods.
- **Writing**: write the conference paper and the M.Sc thesis.

Since the approach adopted is mainly experimental, the steps of design, implementation and testing will be actually iterated many times to find the most promising alternative. At the end, this will be refined and extensively tested.

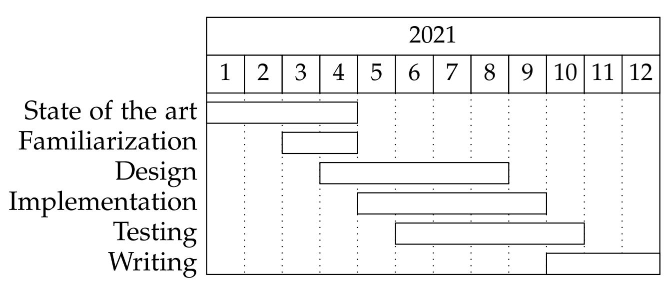

Figure 2: GANTT diagram of the research plan

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