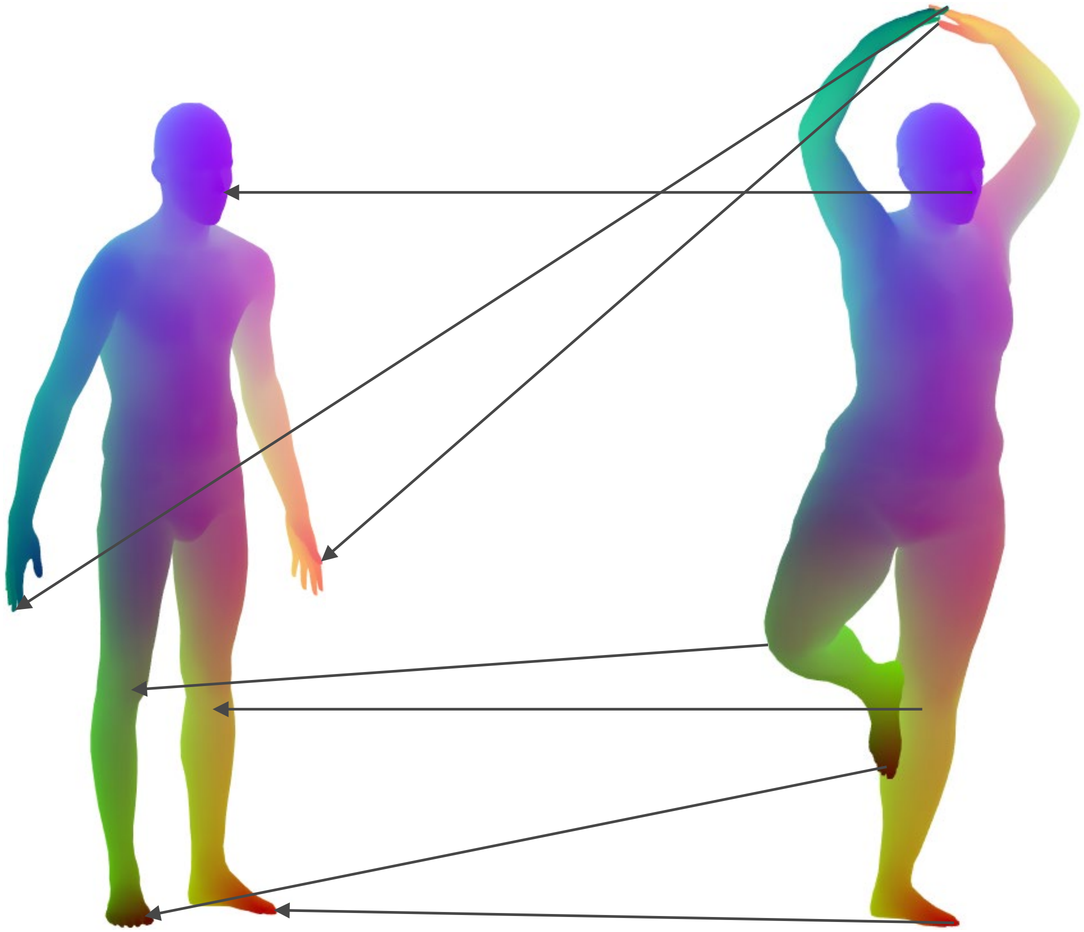
PC-GAU: PCA basis of Scattered Gaussians for Shape Matching via Functional Maps

Michele Colombo
michele11.colombo@mail.polimi.it
CSE Track

Shape Matching: intuitive idea



Find correspondences between the points of two 3D shapes

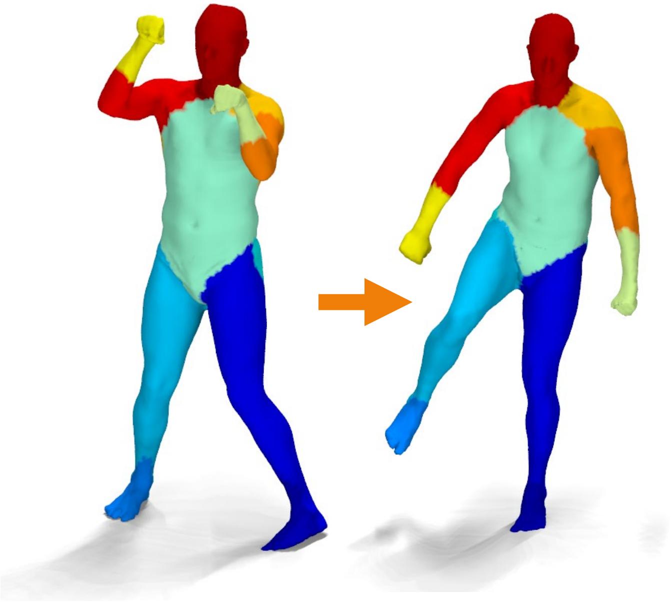
Shape Matching: applications

Transfer of **information** between shapes

- Texture Transfer

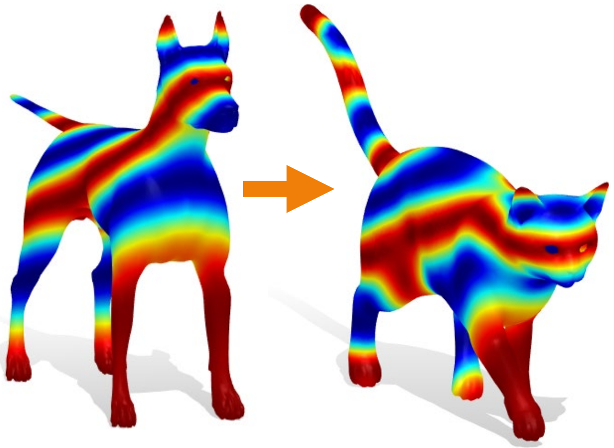
Source: Melzi et al, ACM TOG 2019

- Segmentation transfer



Source: Ovsjanikov et al, ACM TOG 2012

- Function transfer (e.g. deformation)

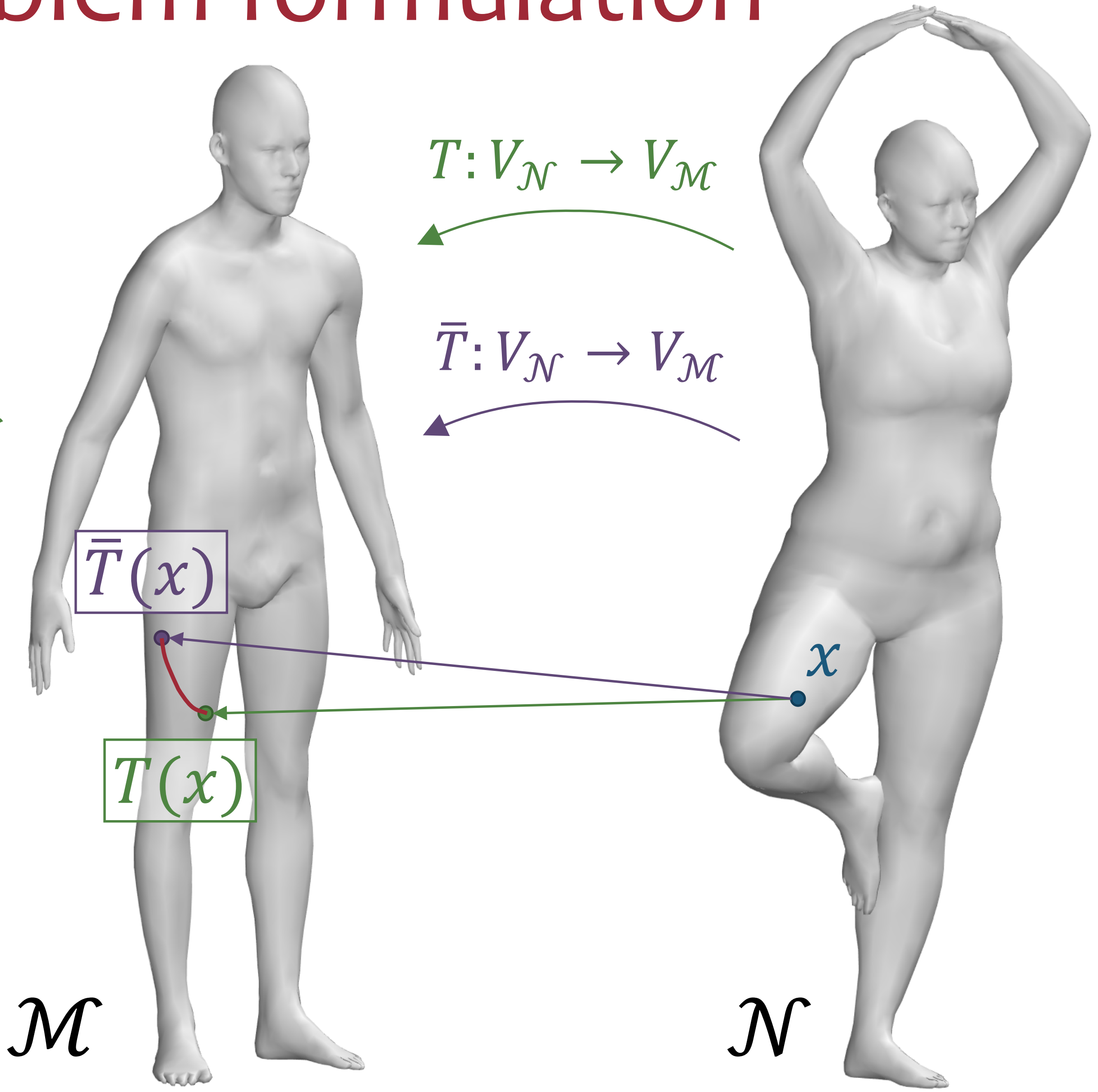


Source: Melzi et al, ACM TOG 2019

Shape Matching: problem formulation

- **Input:** pair of meshes $\mathcal{M}$ and $\mathcal{N}$, vertices $V_{\mathcal{M}}$ and $V_{\mathcal{N}}$
- Unknown correspondence $T: V_{\mathcal{N}} \to V_{\mathcal{M}}$
- **Output:** point-wise map $\bar{T}: V_{\mathcal{N}} \to V_{\mathcal{M}}$
- **Goal:** $\bar{T}$ similar to T
- Error evaluation:

$$e(x) = \text{geoDist}_{\mathcal{M}}(\bar{T}(x), T(x))$$



Shape Matching via Functional Maps [OBCS*12]

Define a functional basis on $\mathcal{M}$ and $\mathcal{N}$

Find functional map $\boxed{C}$

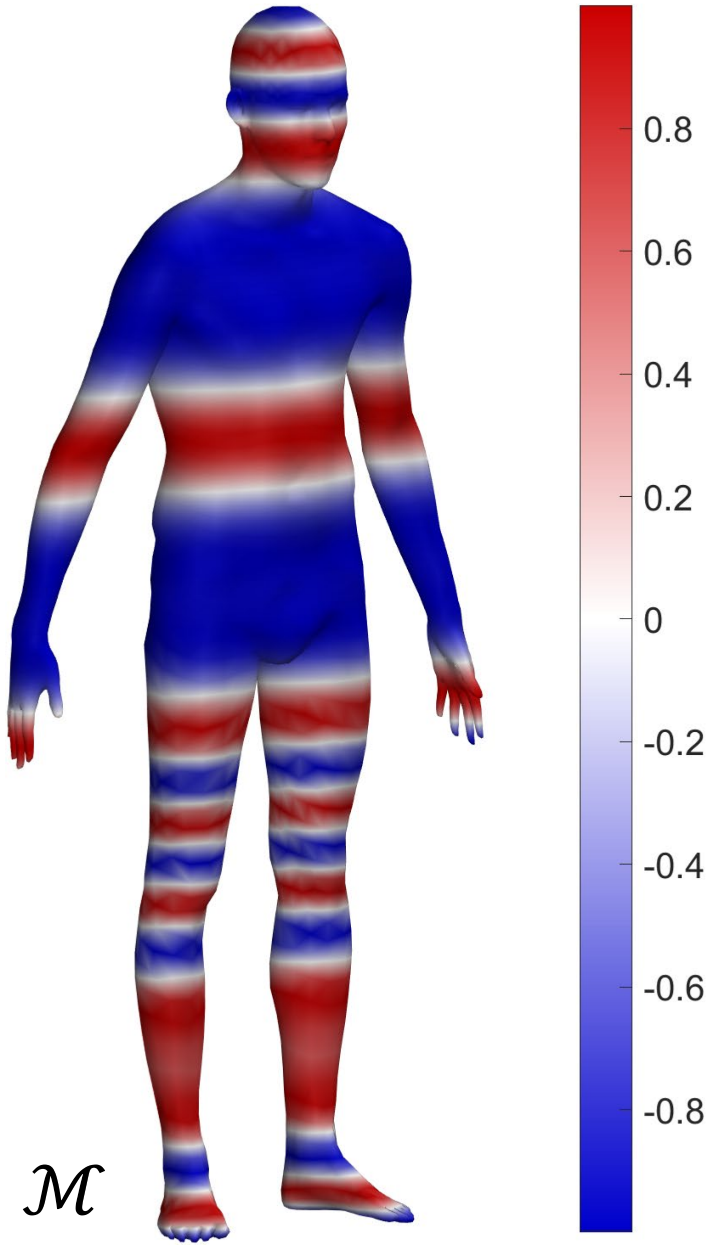
Convert to point-wise map

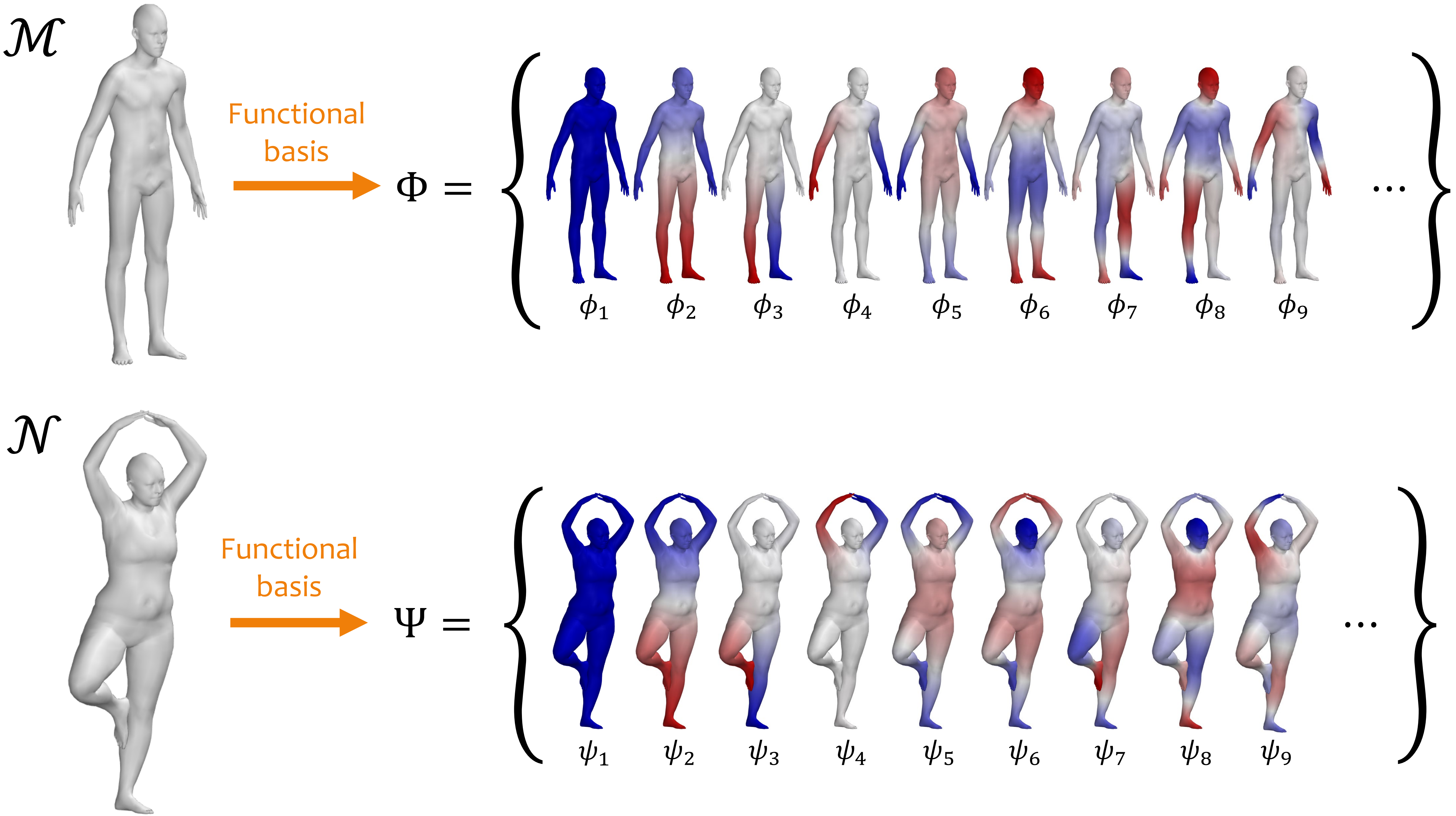
Optimization problem with linear constraints

Compact matrix

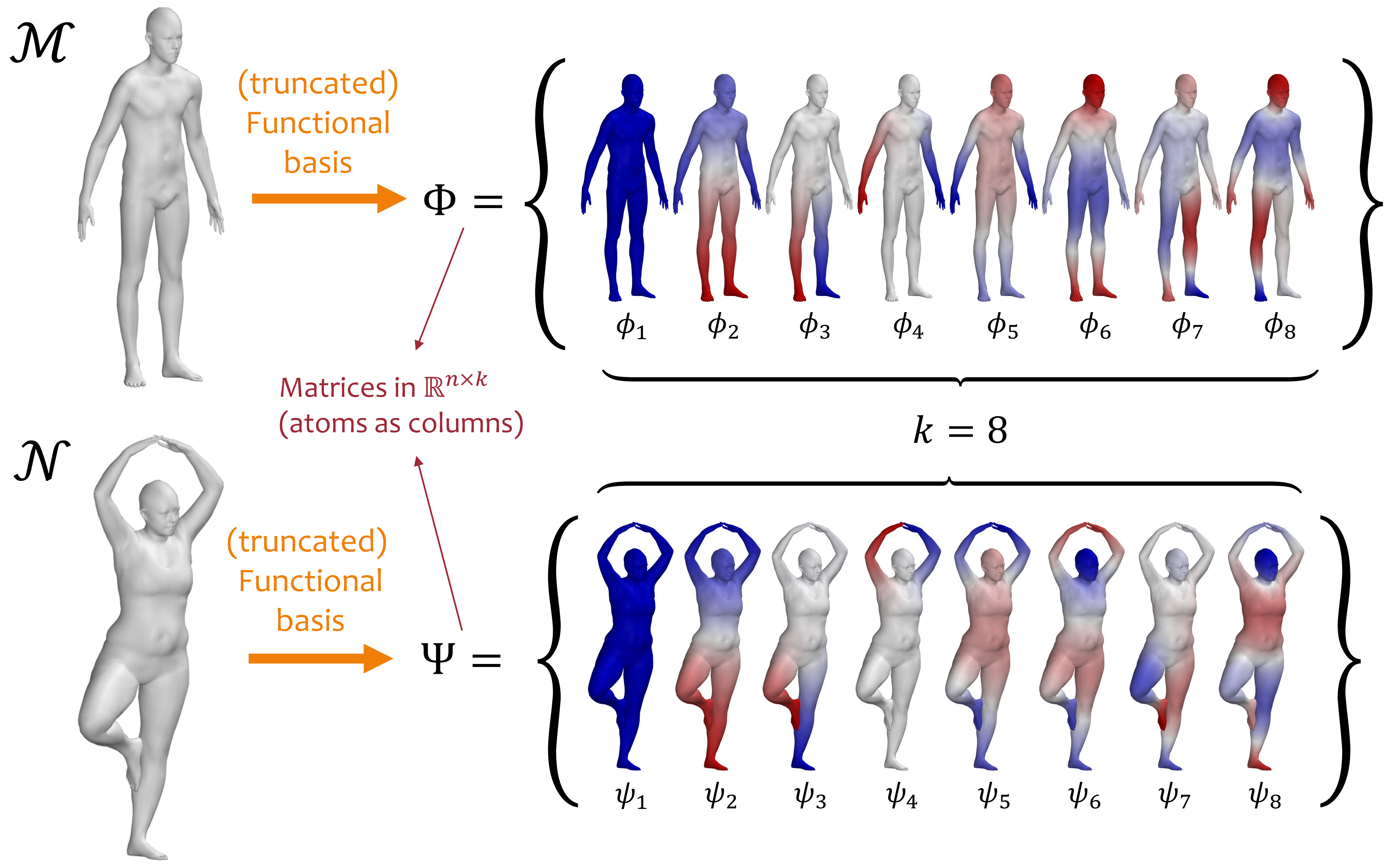
Functions defined on a mesh

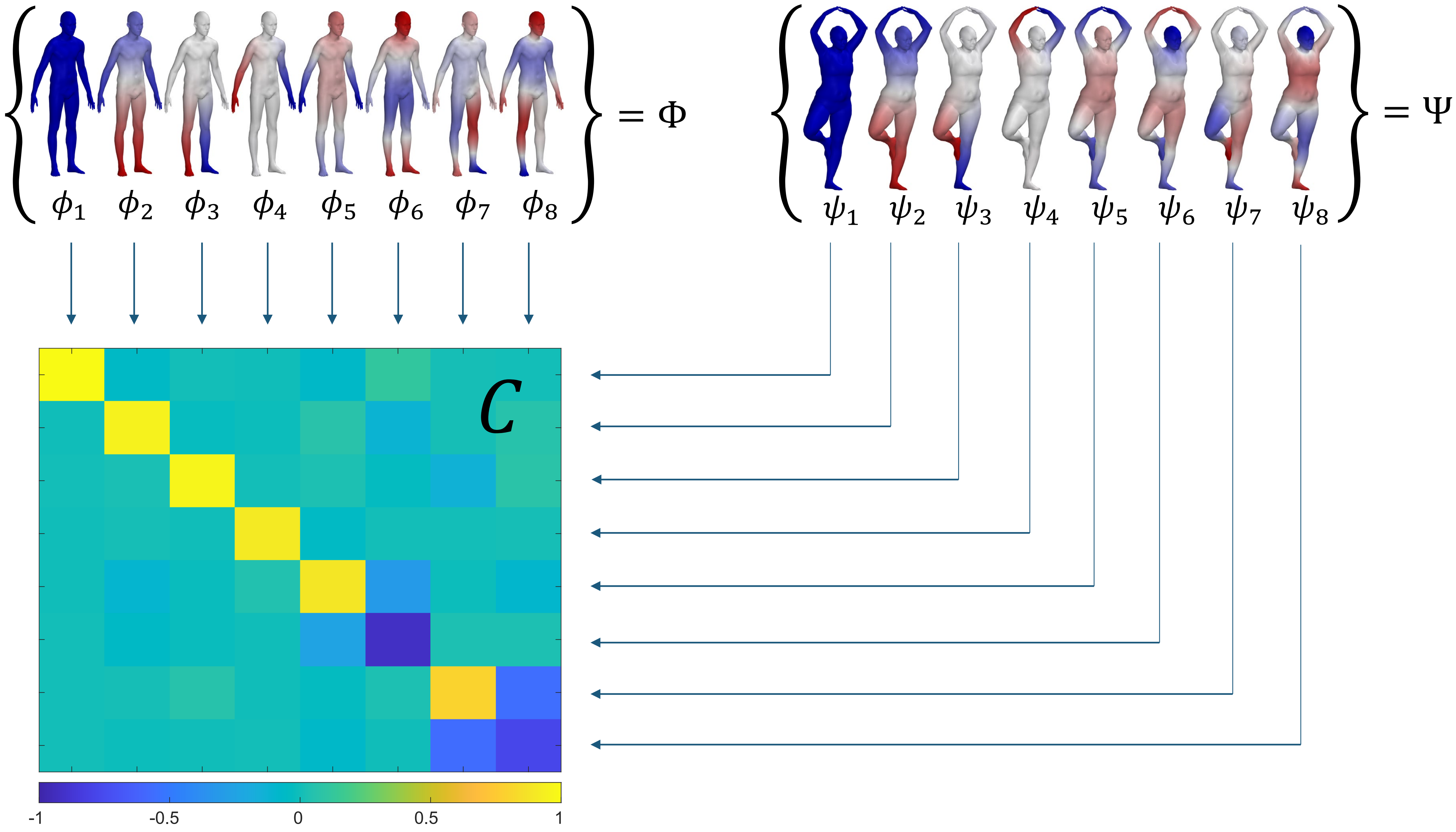
- $f: V_{\mathcal{M}} \to \mathbb{R}$
- f vector $\in \mathbb{R}^n$, with $n = |V_{\mathcal{M}}|$
- $\mathcal{F}(\mathcal{M}, \mathbb{R})$ functional space of $\mathcal{M}$



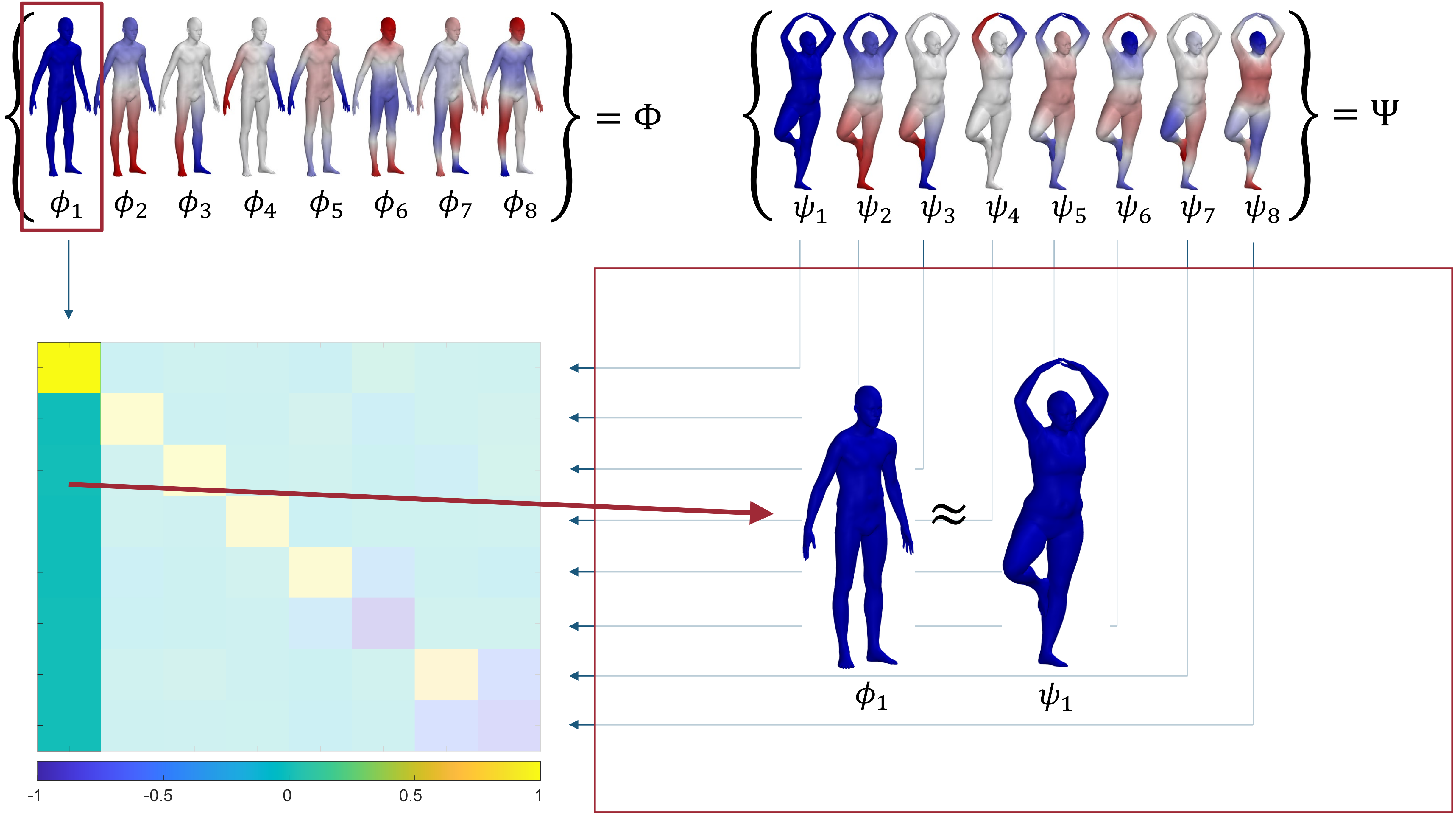


$\mathcal{M}$
Functional basis
$\Phi = \{ \phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4 \quad \phi_5 \quad \phi_6 \quad \phi_7 \quad \phi_8 \quad \phi_9 \quad \cdots \}$
$\mathcal{N}$
Functional basis
$\Psi = \{ \psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \psi_6 \quad \psi_7 \quad \psi_8 \quad \psi_9 \quad \cdots \}$

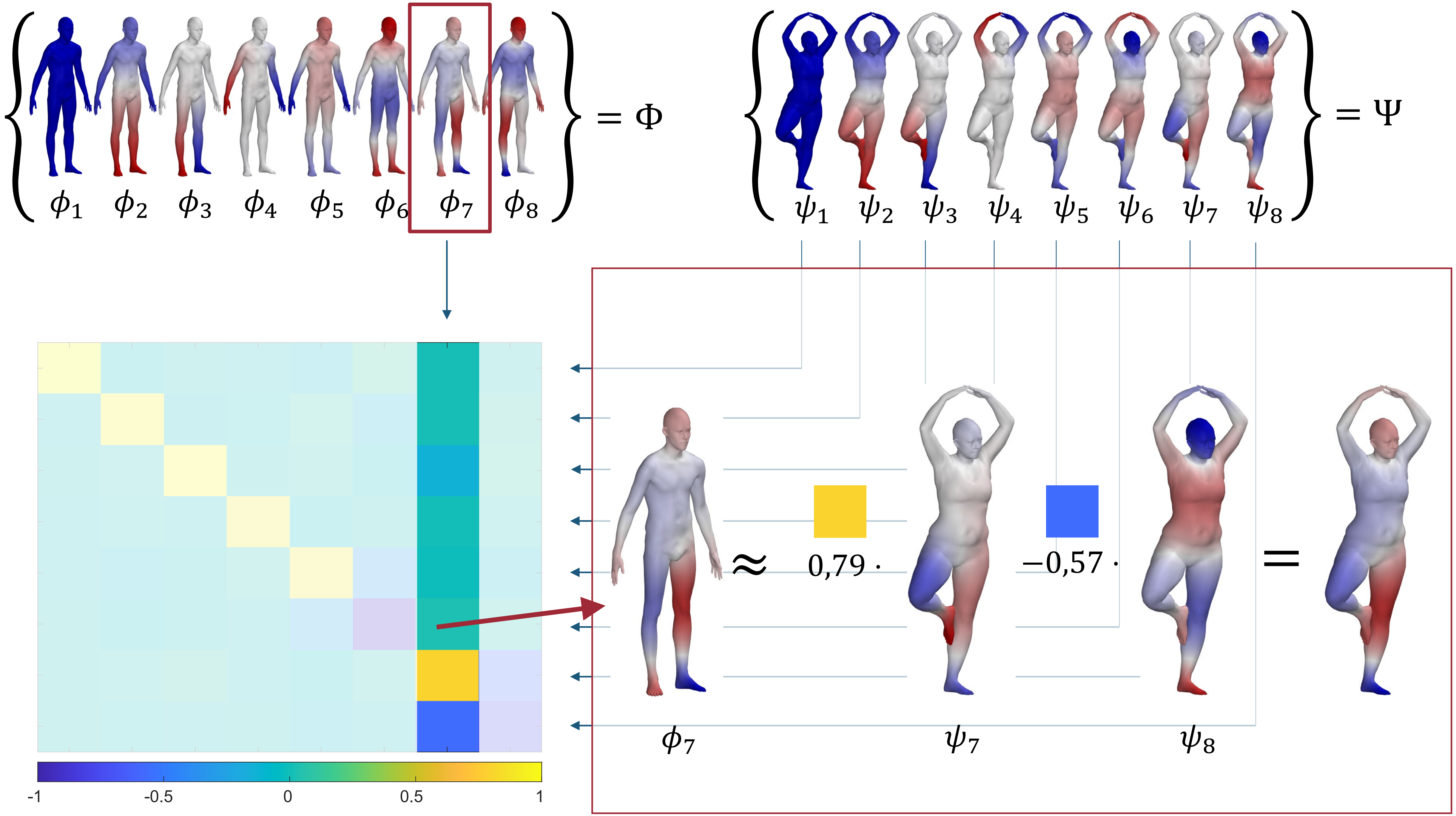

$\mathcal{M}$
(truncated) Functional basis
$\Phi =$
ϕ_1 ϕ_2 ϕ_3 ϕ_4 ϕ_5 ϕ_6 ϕ_7 ϕ_8
Matrices in $\mathbb{R}^{n\times k}$
(atoms as columns)
$k = 8$
$\mathcal{N}$
(truncated) Functional basis
$\Psi =$
ψ_1 ψ_2 ψ_3 ψ_4 ψ_5 ψ_6 ψ_7 ψ_8



ϕ_1 ϕ_2 ϕ_3 ϕ_4 ϕ_5 ϕ_6 ϕ_7 ϕ_8
$= \Phi$
ψ_1 ψ_2 ψ_3 ψ_4 ψ_5 ψ_6 ψ_7 ψ_8
$= \Psi$
C
-1
-0.5
0
0.5
1



ϕ_1 ϕ_2 ϕ_3 ϕ_4 ϕ_5 ϕ_6 ϕ_7 ϕ_8
$= \Phi$
ψ_1 ψ_2 ψ_3 ψ_4 ψ_5 ψ_6 ψ_7 ψ_8
$= \Psi$
$\phi_1 \approx \psi_1$
-1
-0.5
0
0.5
1


ϕ_1 ϕ_2 ϕ_3 ϕ_4 ϕ_5 ϕ_6 ϕ_7 ϕ_8
$= \Phi$
ψ_1 ψ_2 ψ_3 ψ_4 ψ_5 ψ_6 ψ_7 ψ_8
$= \Psi$
$\approx$ $0{,}79 \cdot$ $-0{,}57 \cdot$ $=$
ϕ_7 ψ_7 ψ_8
-1 -0.5 0 0.5 1

Function transfer through C

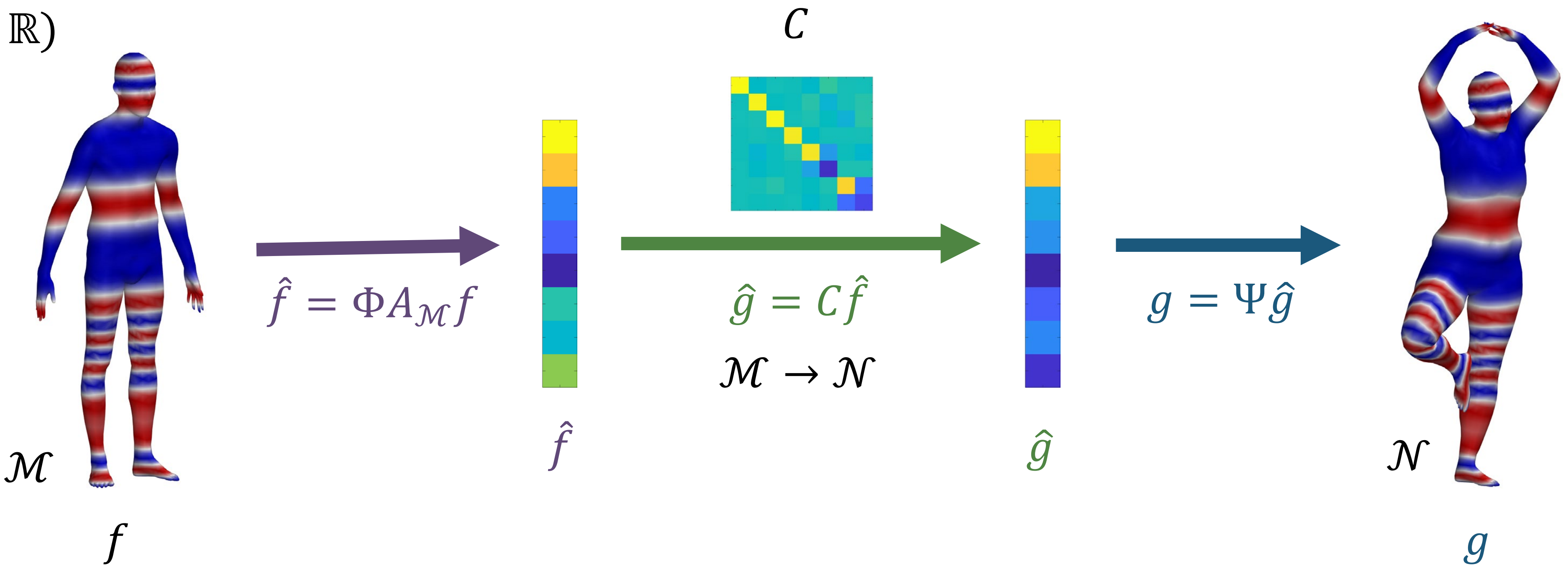
$$g = \underbrace{\Psi \underbrace{C \underbrace{\Phi A_{\mathcal{M}} f}_{\hat{f}}}_{\hat{g} = C\,\hat{f}}}_{g = \Psi \hat{g}}$$

$\hat{f} \rightarrow$ Coefficients of f on Φ

$\hat{g} = C\,\hat{f} \rightarrow$ Coefficients transferred on $\mathcal{N}$

$g = \Psi \hat{g} \rightarrow$ Reconstruction of g from coefficients $\hat{g}$

- $\mathcal{M}$ with (truncated) orthonormal basis Φ
- $\mathcal{N}$ with (truncated) orthonormal basis Ψ
- $C \in \mathbb{R}^{k \times k}$
- $f \in \mathcal{F}(\mathcal{M}, \mathbb{R})$



Finding C

For each pair of corresponding functions f_i and g_i

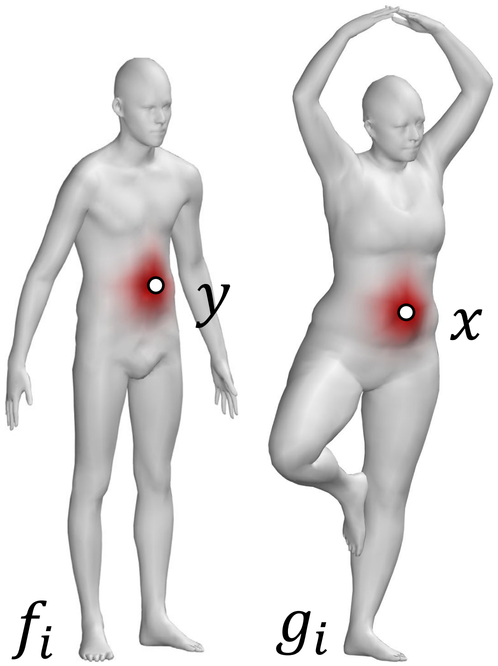
Projection on Φ and Ψ

$$\hat{f}_i = \Phi A_{\mathcal{M}} f_i \qquad \hat{g}_i = \Psi A_{\mathcal{N}} g_i$$

Preservation -> Constraint on C

$$\hat{g}_i = C\hat{f}_i$$

Optimization problem on C



Landmark: $y = T(x)$

Descriptors

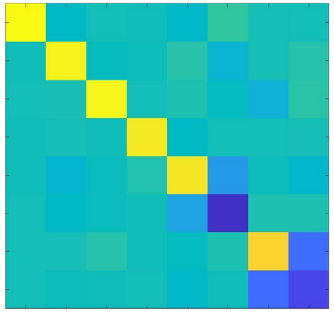
Linear in C

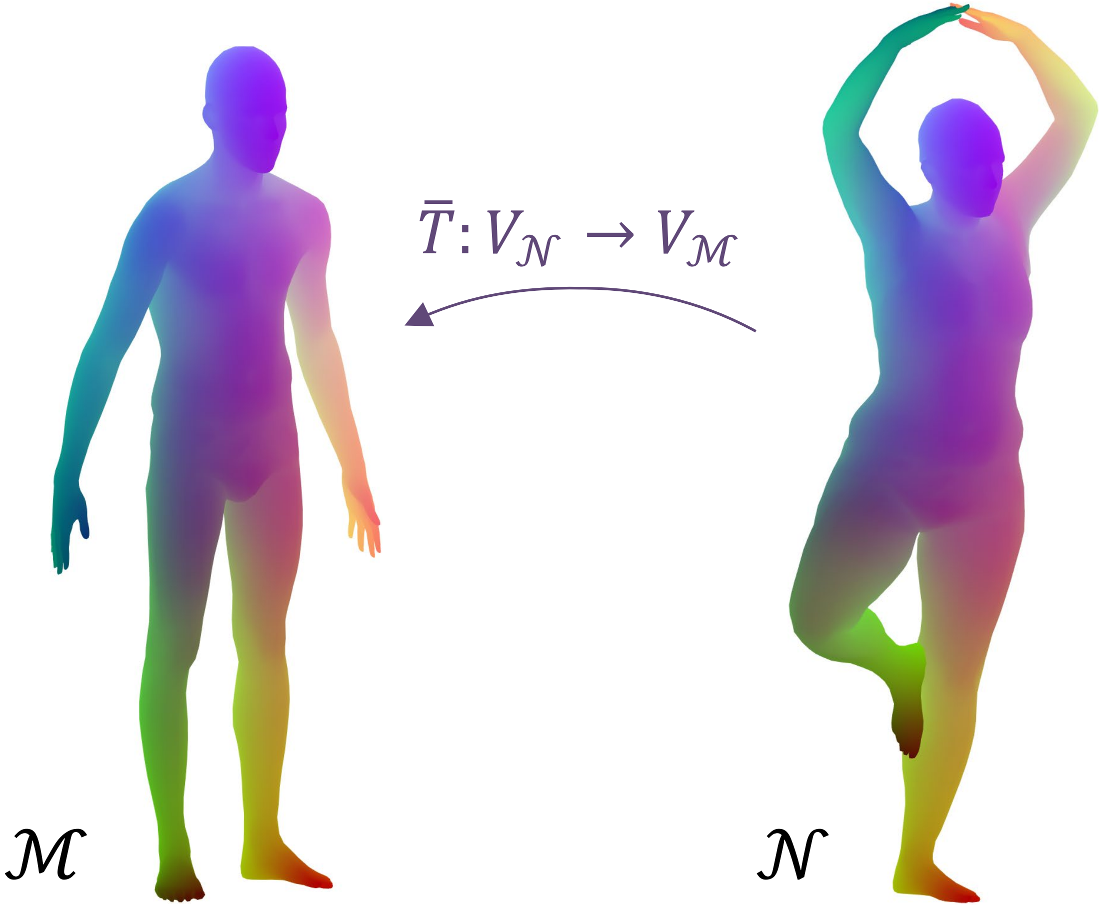
$$\boxed{\min \sum_i \left\| \hat{g}_i - C\hat{f}_i \right\|_2^2}$$

Converting C into a point-wise map [OBCS*12]

Given:

- $\mathcal{M}$ with basis Φ
- $\mathcal{N}$ with basis Ψ
- C from $\mathcal{M}$ to $\mathcal{N}$

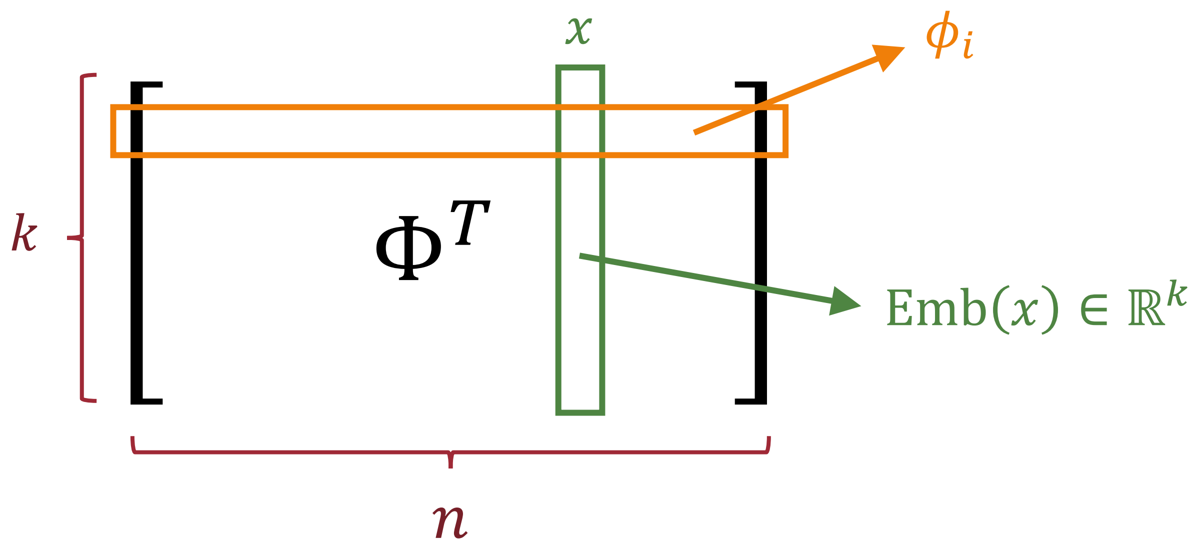




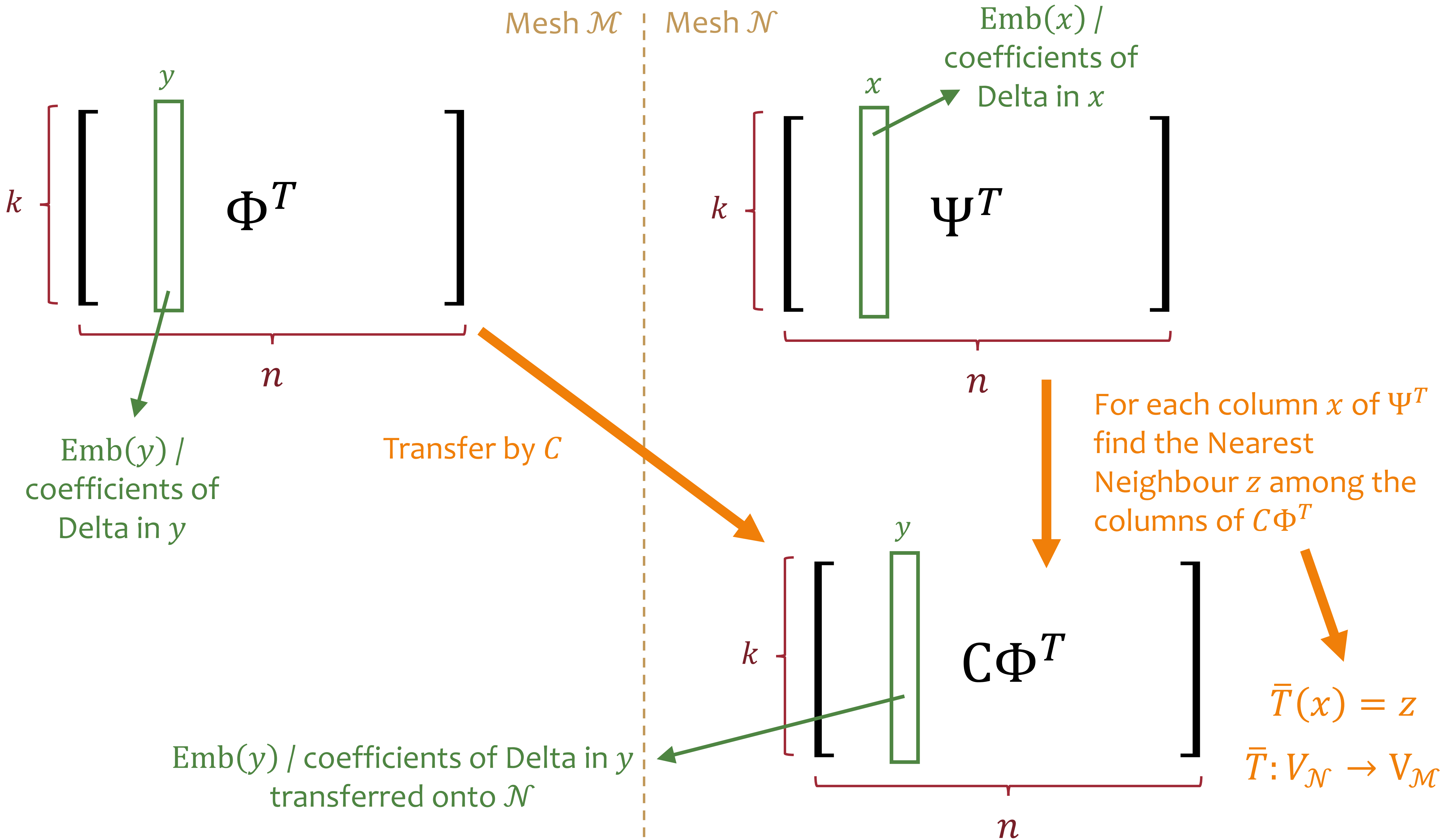
Embedding

- Mesh $\mathcal{M}$, basis $\Phi = \{\phi_i\}$, vertex $x \in V_{\mathcal{M}}$
- Embedding of x:

$$\mathrm{Emb}\,(x) = [\phi_i(x)]$$



- k-dimensional representation of vertex x
- $\mathrm{Emb}(x)$ are the coefficients in Φ of a Delta function centered in x



Mesh $\mathcal{M}$
Mesh $\mathcal{N}$
y
k
Φ^T
n
Emb(y) / coefficients of Delta in y
Transfer by C
x
Emb(x) / coefficients of Delta in x
k
Ψ^T
n
For each column x of Ψ^T find the Nearest Neighbour z among the columns of $C\Phi^T$
y
k
$C\Phi^T$
n
$\bar{T}(x) = z$
$\bar{T}: V_{\mathcal{N}} \rightarrow V_{\mathcal{M}}$
Emb(y) / coefficients of Delta in y transferred onto $\mathcal{N}$

Locality preservation

- This method works if:

$$\|\mathrm{Em}b(x) - \mathrm{Emb}(y)\|_2 \approx \mathrm{geoDist(x,y)}\ \forall x, \mathrm{y} \in \mathcal{M}$$

- In particular, **ordering** of $\|\mathrm{Em}b(x) - \mathrm{Emb}(y)\|$ and geoDist(x,y) for all vertices y in a neighborhood of x should be preserved

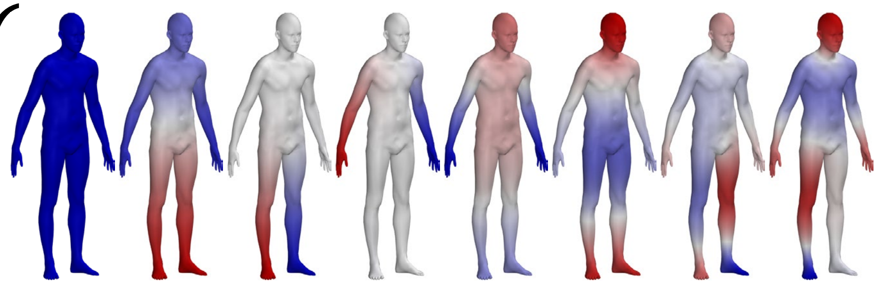
We will define a metric for this property

Φ should be **locality preserving** in $x, \forall x \in \mathcal{M}$

Standard basis: LB

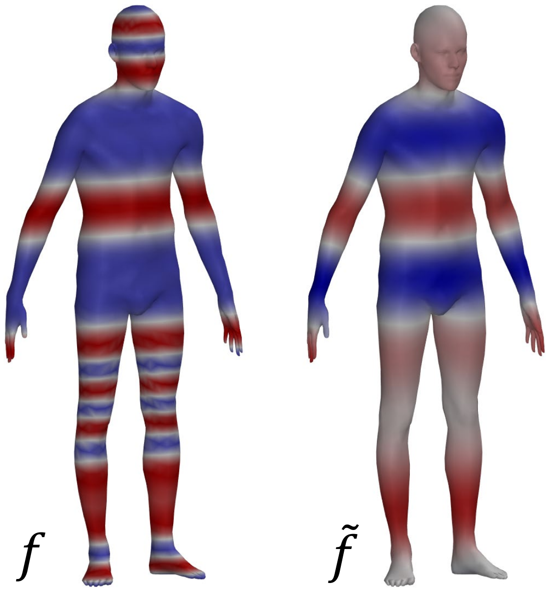
Eigenfunctions of the Laplace-Beltrami operator: $\Delta\phi_i = \lambda_i\phi_i$ [VL08]

$\Phi_{\mathrm{LB}} = \{$



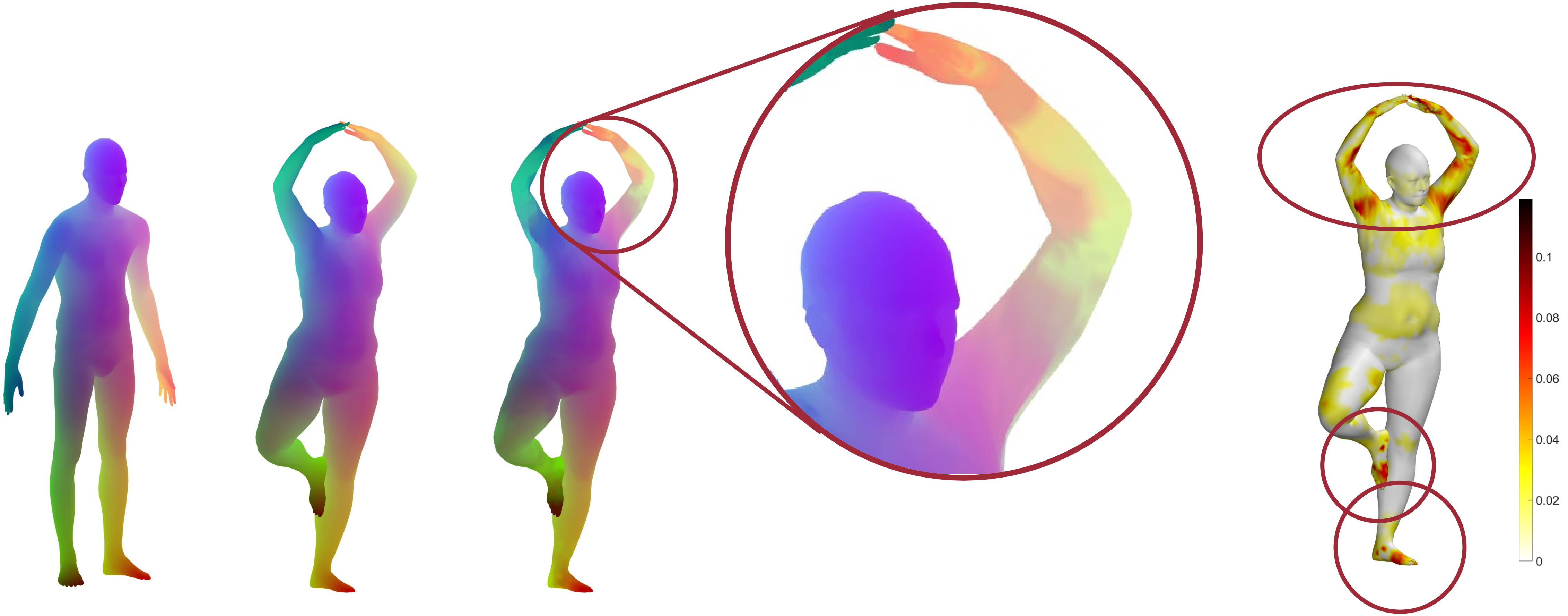
$\cdots \}$

- Orthonormal ⇒ efficient projection
- Ordered in frequency ⇒ low-pass filter approximation of functions
- Optimal k-dimensional basis for approximating smooth functions on a mesh [ABK15]



$$\tilde{f} = \Phi_{LB}\Phi_{LB}^{T}A_{\mathcal{M}}f$$

Errors in point-wise maps



ground truth T Estimated $\bar{T}$ Error: $e(x)$

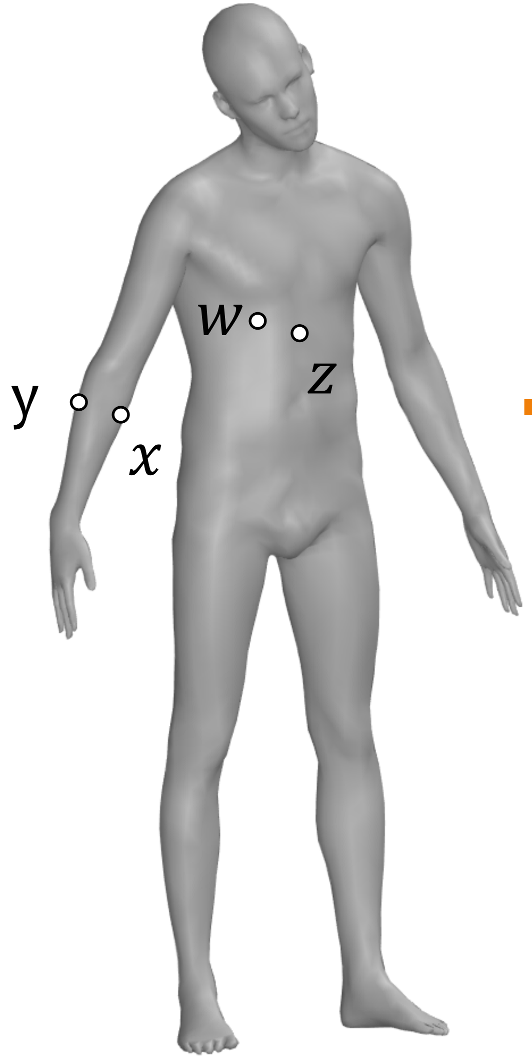
Limits of LB

The energy of LB is **not distributed evenly** on the mesh surface

- Low quality of representation provided by the embedding space in certain areas:
  - Discrimination power between different vertices
  - Locality preservation

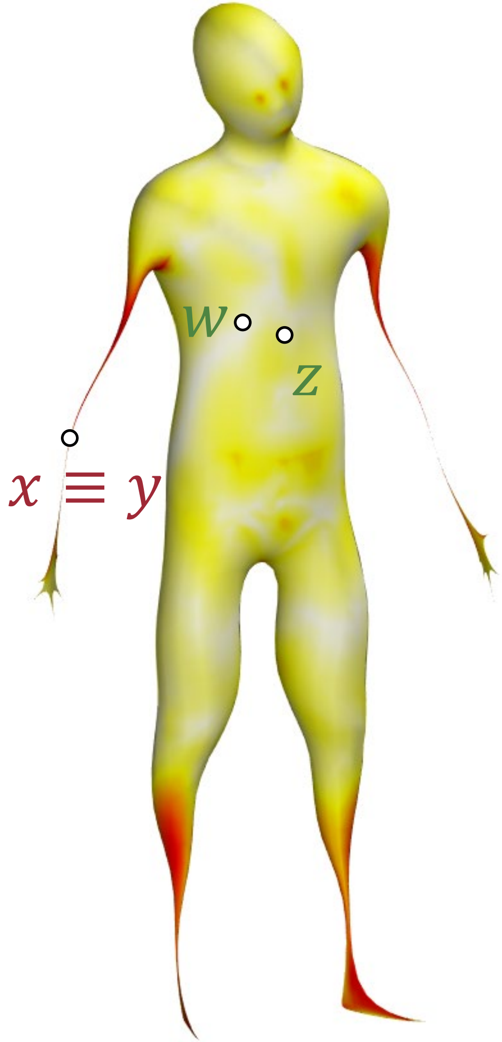
We will define metrics

Energy distribution: intuition



LB approximation

$$\tilde{X} = \Phi\Phi^T A_{\mathcal{M}} X$$
$$\tilde{Y} = \Phi\Phi^T A_{\mathcal{M}} Y$$
$$\tilde{Z} = \Phi\Phi^T A_{\mathcal{M}} Z$$

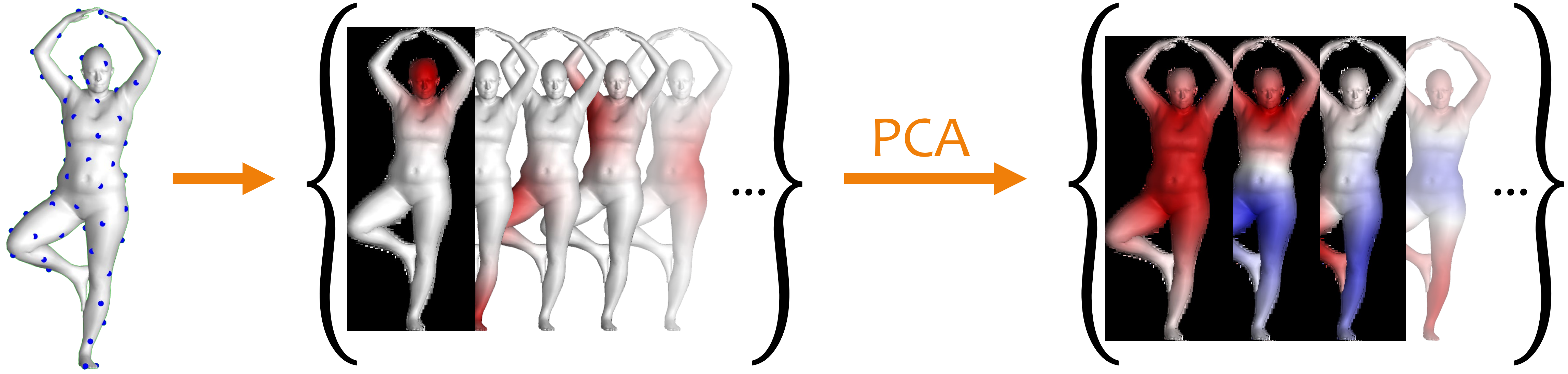


Our proposal: PC-GAU

- Basis for $\mathcal{F}(\mathcal{M}, \mathbb{R})$
- Designed to be used as a truncated basis in **functional maps** pipelines for shape matching
- Its energy is **evenly distributed** on the mesh surface

PC-Gau: construction

- Principal Components of a dictionary of Gaussian functions scattered on the mesh

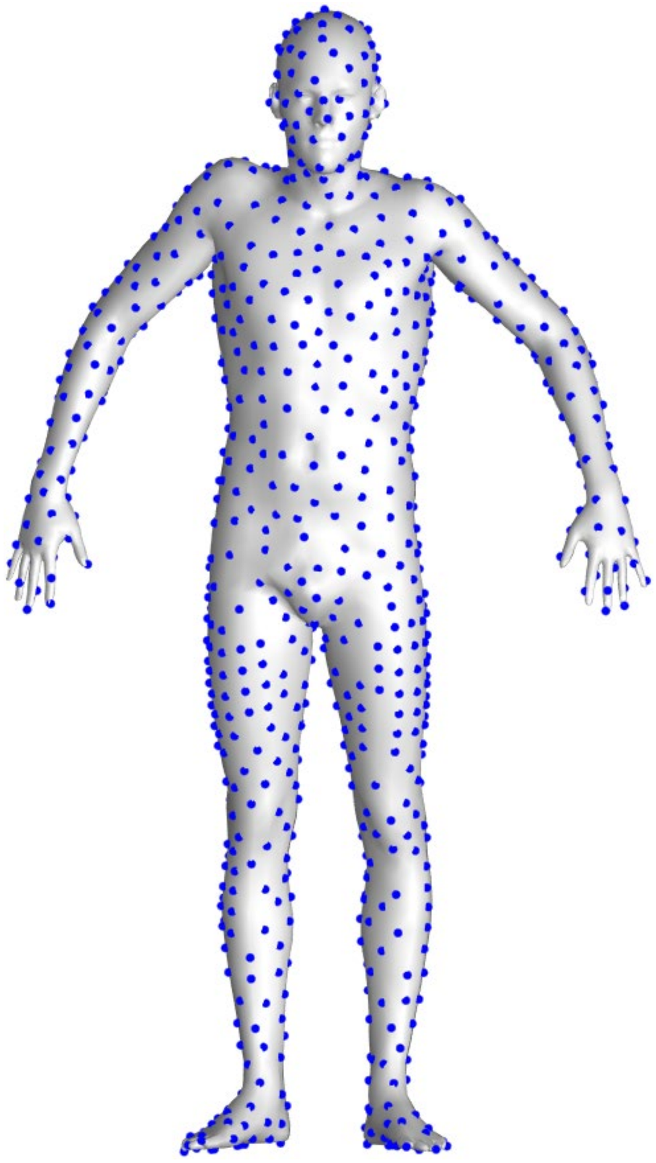


Scattered vertices | Dictionary of Gaussians | PC-Gau

Subset of vertices Q

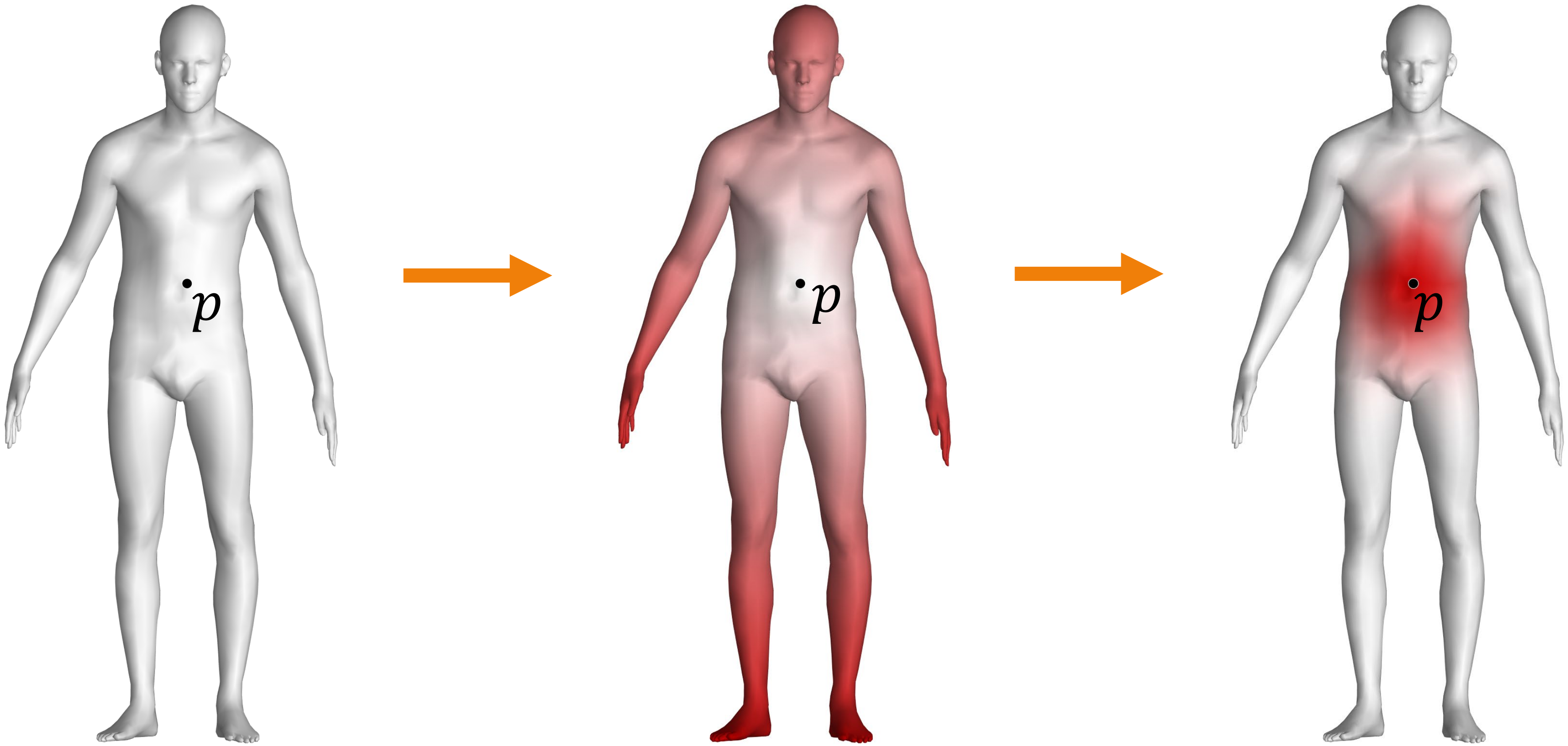
- Subset of q vertices $Q \subset V_{\mathcal{M}}$
- Farthest Point Sampling (Euclidean distance)

Evenly scattered on the surface



$q = 1000$

Gaussian function centered in $p \in Q$



Parameter σ:
Gaussian
amplitude

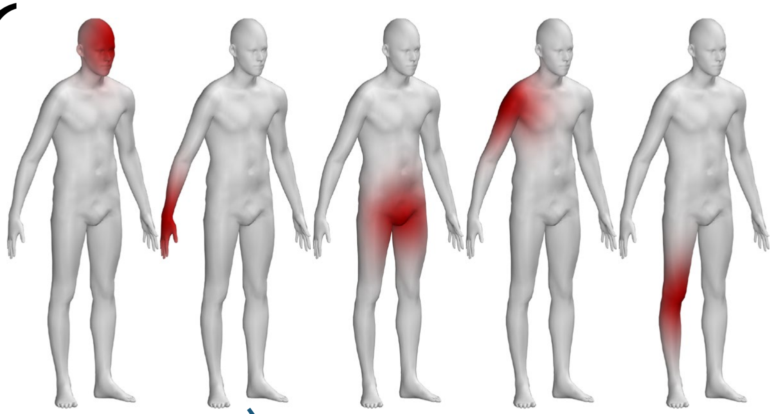
$$D_p(x) = \mathrm{geoDist}_{\mathcal{M}}(p, x)$$
$$x \in V_{\mathcal{M}}$$

$$G_p(x) = \exp(D(x)^2 \boxed{/\sigma})$$
$$x \in V_{\mathcal{M}}$$

Dictionary of Gaussians

A Gaussian centered in each $p \in Q$

Dictionary of q Gaussians $\longrightarrow$ $\boxed{G} = \{$



$\dots \}$

Matrix $G \in \mathbb{R}^{n \times q}$
(Gaussians as columns)

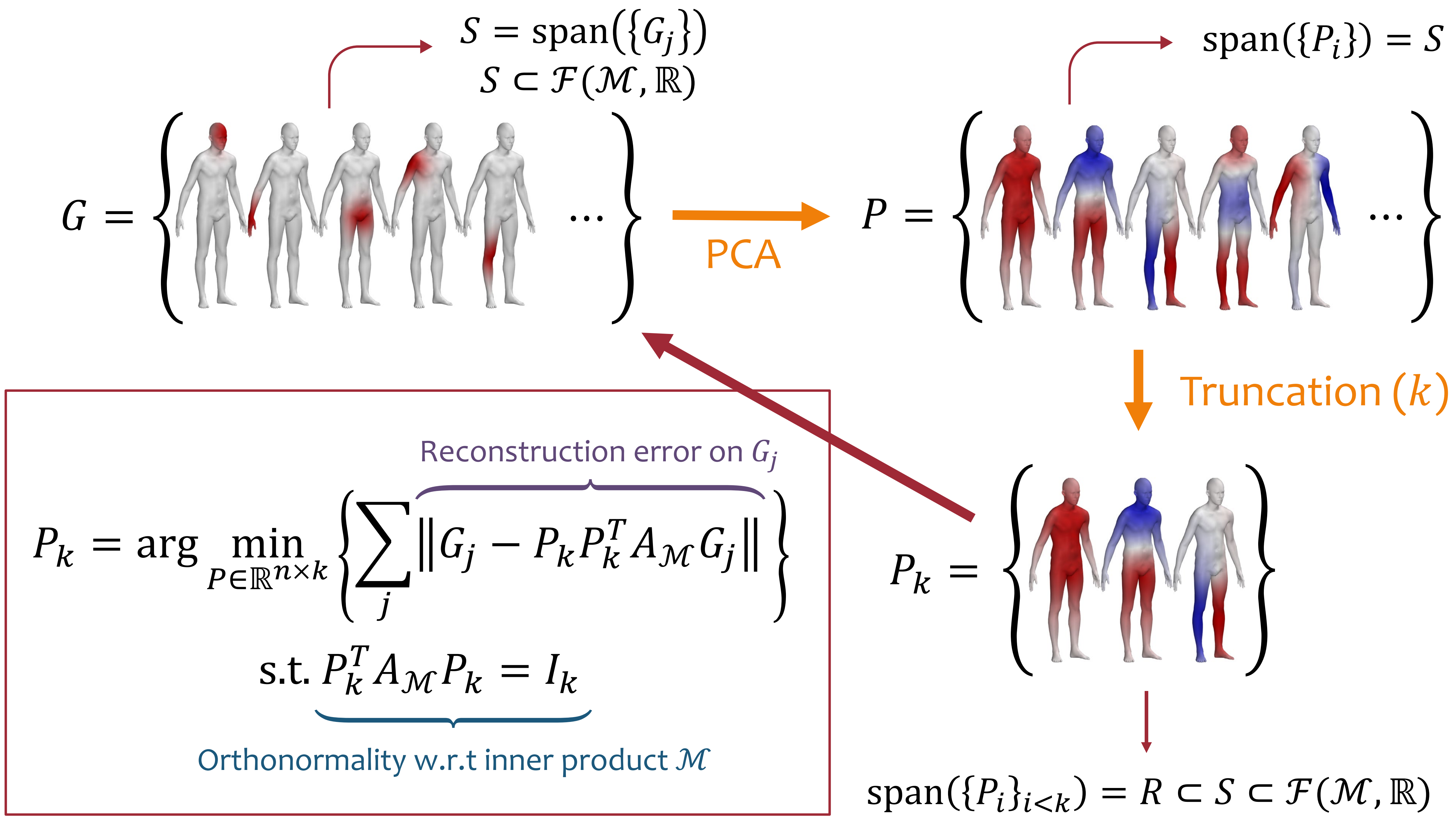
$$\begin{bmatrix} \quad & G & \quad \end{bmatrix}$$

PCA computation

Variable: vertex i

Sample: Gaussian G_j

$$\mathrm{PCA}\left(\underbrace{\left[\qquad G^T \qquad\right]}_{n \text{ vertices}}\right) \quad \to \; q \text{ Gaussians}$$



$S = \mathrm{span}(\{G_j\})$
$S \subset \mathcal{F}(\mathcal{M}, \mathbb{R})$
$\mathrm{span}(\{P_i\}) = S$
$G =$
PCA
$P =$
Truncation (k)
Reconstruction error on G_j
$P_k = \arg\min_{P \in \mathbb{R}^{n \times k}} \left\{ \sum_j \| G_j - P_k P_k^T A_{\mathcal{M}} G_j \| \right\}$
s.t. $P_k^T A_{\mathcal{M}} P_k = I_k$
Orthonormality w.r.t inner product $\mathcal{M}$
$P_k =$
$\mathrm{span}(\{P_i\}_{i<k}) = R \subset S \subset \mathcal{F}(\mathcal{M}, \mathbb{R})$

Even distribution of basis energy

Basis energy -> power of representation of the embedding space

- Discrimination power → $\mathrm{Dis}(x)$
- Locality preservation → $\mathrm{EGDC}(x)$

Discrimination power

Assign sufficiently distant embeddings to different vertices

Function on $\mathcal{M}$

$$\boxed{Dis(x)} = \frac{\|Emb(x) - Emb(y)\|_2}{geoDist(x, y)}$$

Nearest neighbor
in embedding space

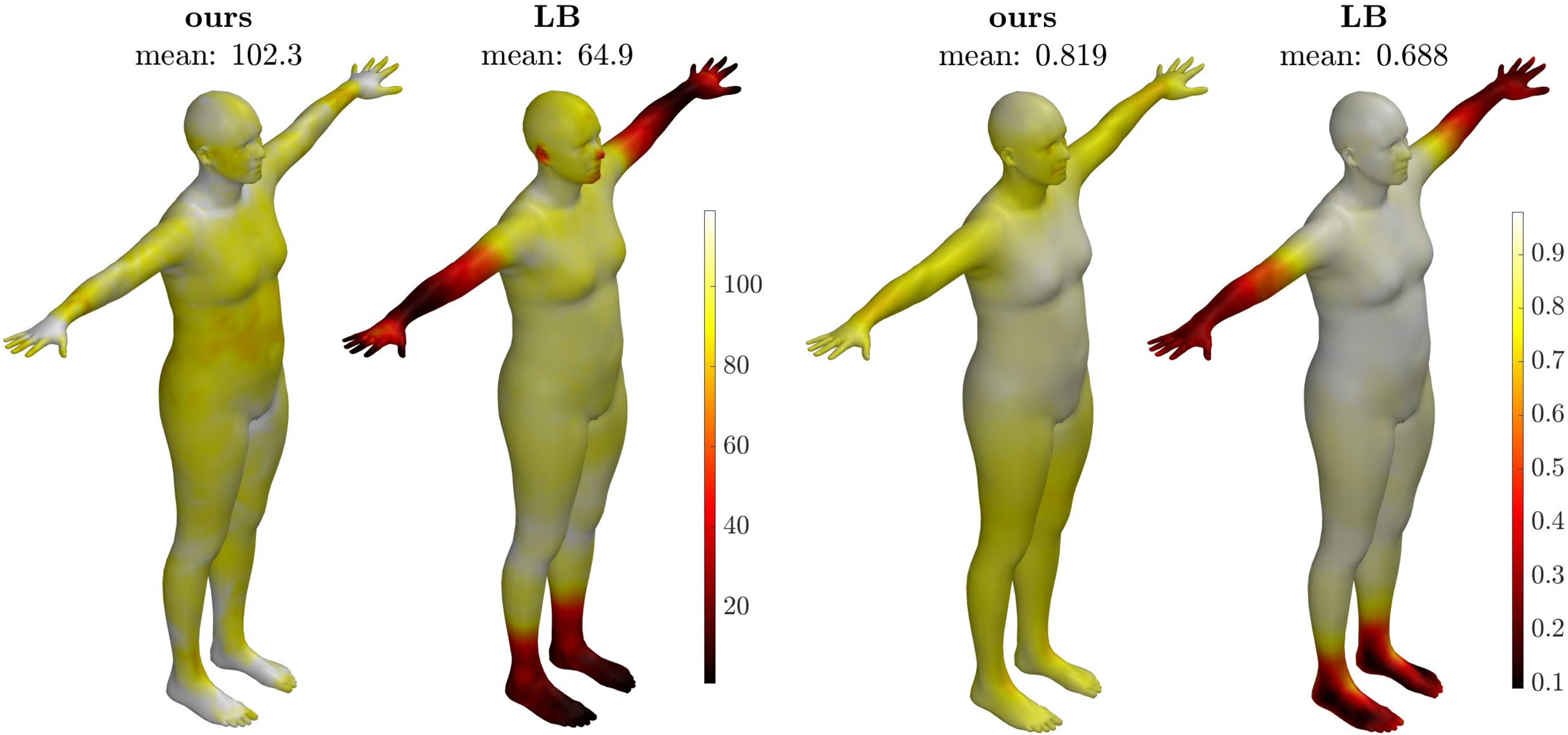
$$\boxed{y} = \arg \min_{z \in V_{\mathcal{M}} \setminus \{x\}} \{\|Emb(z) - Emb(x)\|_2\}$$

Locality preservation

Correlation to measure the order preservation between embedding and geodesic distances

$$EGDC(x) = corr\left(geoDist_{\mathcal{M}}(x,y), \|Emb(x) - Emb(y)\|_2\right)_{y\in E}$$

E set of s embeddings closest to $Emb(x)$



Discrimination power $Dis(x)$

Locality preservation $EGDC(x)$

Expressive power of the basis

Better (more even) distribution

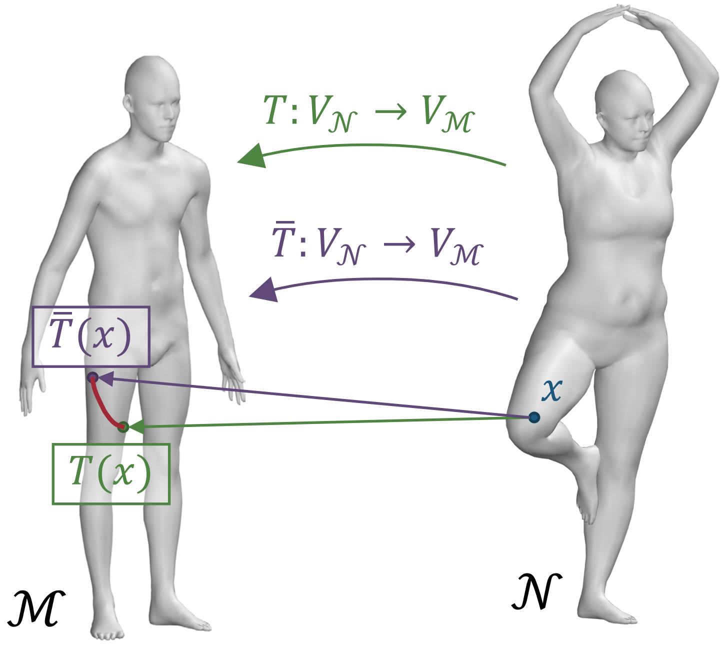
Better overall

Better accuracy in point-wise maps ?

Experimental Evaluation

- Randomly select pairs of meshes from established datasets
- Insert LB and PC-GAU in the same shape matching pipeline
- Compare the accuracy of the point-wise maps obtained

Point-wise accuracy



- Evaluate $\bar{T}: V_{\mathcal{N}} \to V_{\mathcal{M}}$ wrt $T: V_{\mathcal{N}} \to V_{\mathcal{M}}$ (given)
- Geodesic error:

$$e(x) = \text{geoDist}_{\mathcal{M}}(T(x), \bar{T}(x))$$

Average Geodesic Error (AGE) → Overall accuracy

Cumulative error curves → % Correspondences Within error threshold

Plot on mesh → Spatial distribution

C computed from the ground-truth correspondence

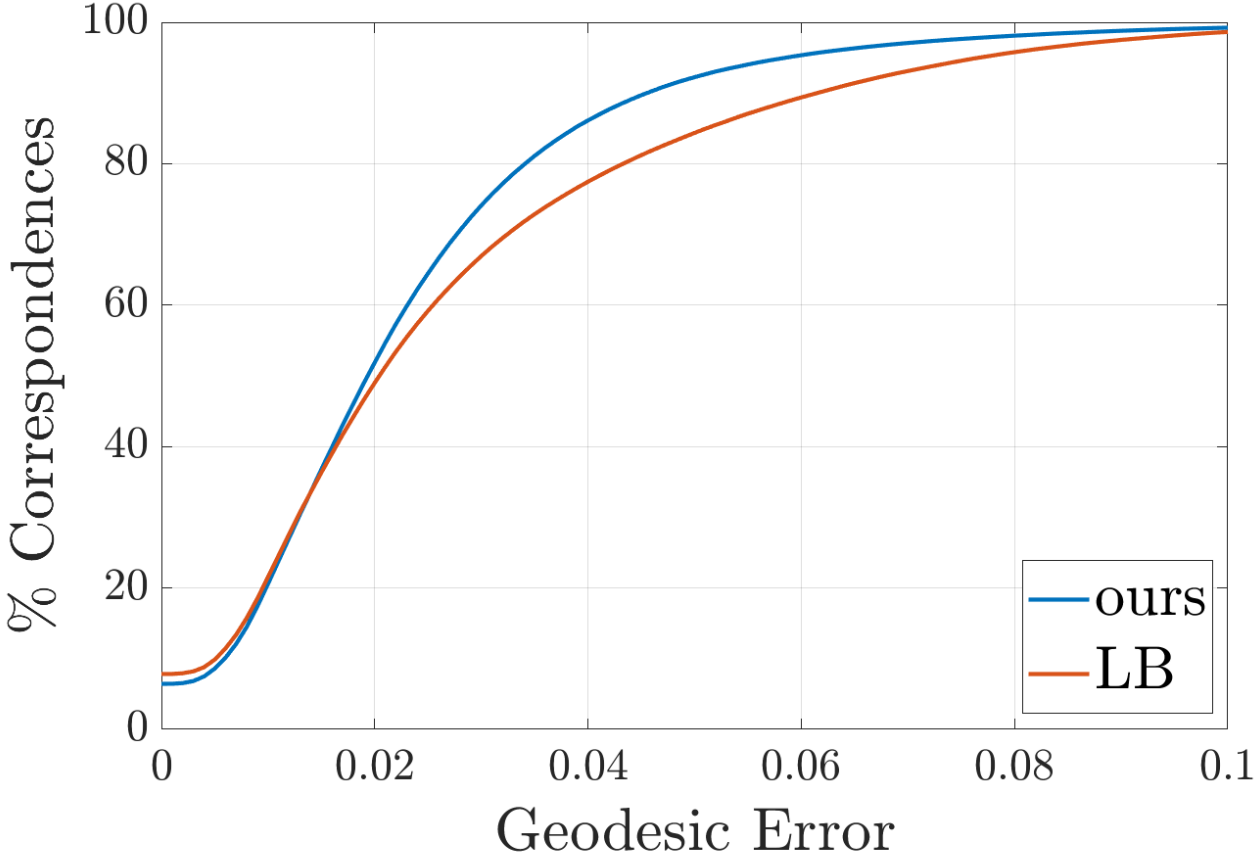
- Test setting only
- Best possible C, given Φ and Ψ

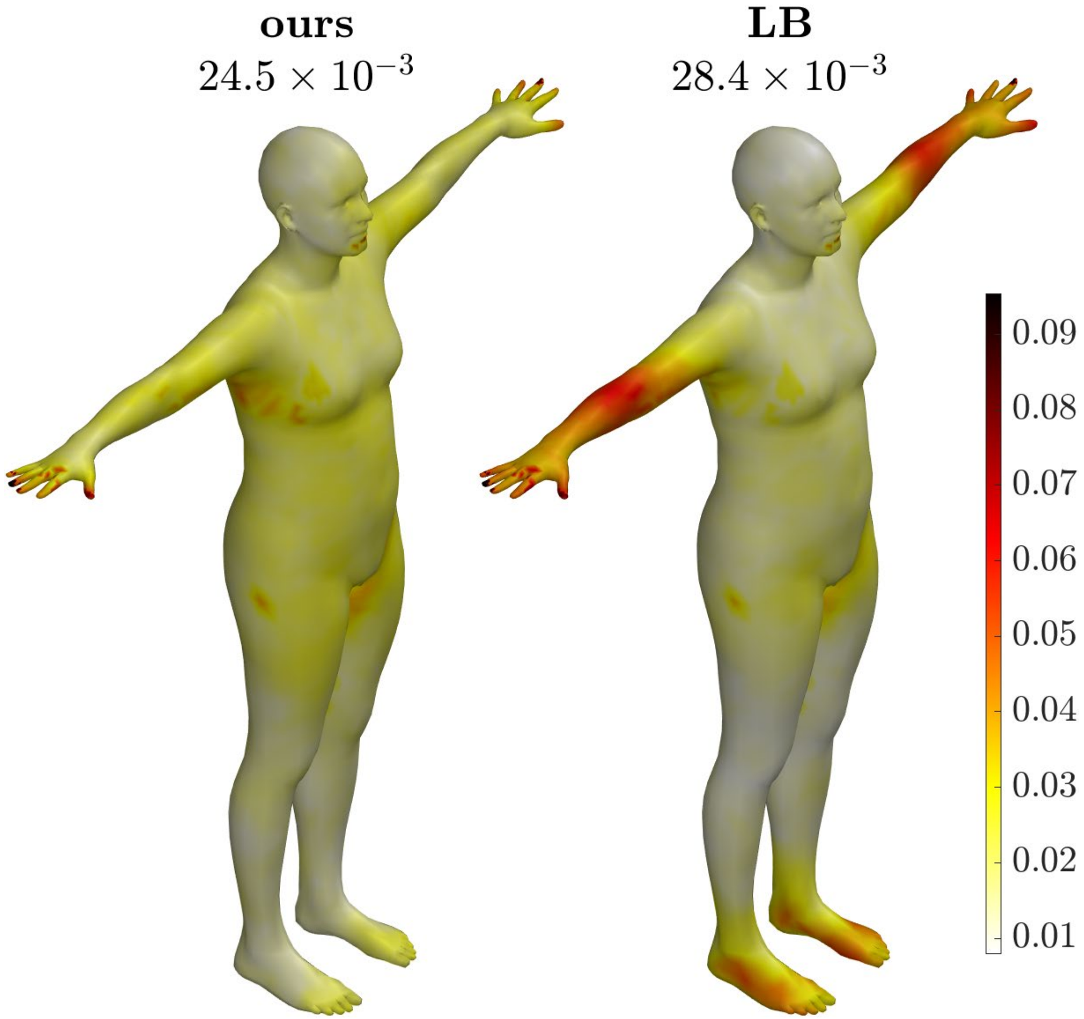
Average Geodesic Error

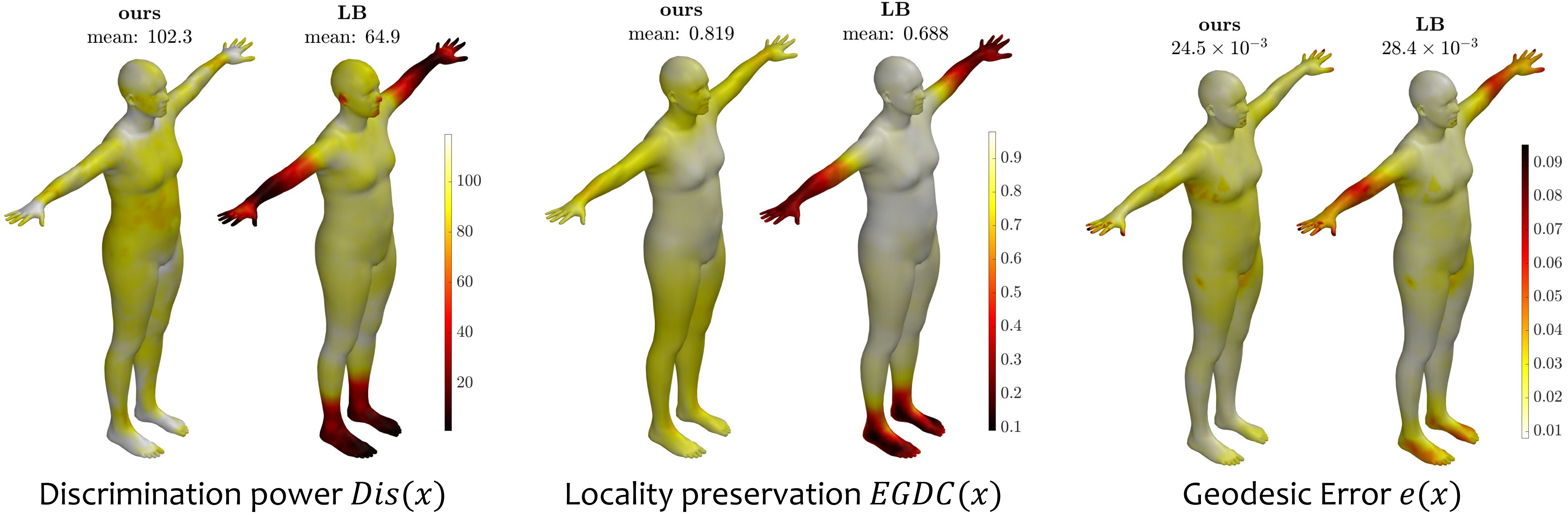
Dataset	Ours ($\times 10^{-3}$)	LB ($\times 10^{-3}$)
FAUST	**15,7**	19,7
MWG	**20,8**	24,9
MWG iso	**13,6**	17,3
TOSCA	**7,7**	12,3
SHREC19	**24,5**	28,4

SHREC19 dataset

AGE: Ours: **24,5** – LB: 28,4







ours
mean: 102.3
LB
mean: 64.9
100
80
60
40
20
Discrimination power $Dis(x)$
ours
mean: 0.819
LB
mean: 0.688
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
Locality preservation $EGDC(x)$
ours
24.5×10^{-3}
LB
28.4×10^{-3}
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
Geodesic Error $e(x)$

C estimated in real settings

	Average Geodesic Error			
Dataset	**Ours** $(\times 10^{-3})$	**LB** $(\times 10^{-3})$	**Ours** $(\times 10^{-3})$	**LB** $(\times 10^{-3})$
FAUST	**28,0**	30,6	**24,0**	26,1
MWG	**58,7**	**58,7**	**51,2**	70,6
MWG iso	**25,9**	27,6	**18,6**	27,2
TOSCA	**12,7**	19,8	**9,7**	20,5
SHREC19	**56,2**	78,5	**34,5**	39,4
	C estimated with product preservation [NO17]		C estimated with ZoomOut [MRR*19]	

Expressive power of the basis

Better (more even) distribution

Better overall

Relation

Better accuracy in point-wise maps

References

[ABK15] Y. Aflalo, H. Brezis, and R. Kimmel. On the optimality of shape and data representation in the spectral domain. *SIAM J. Imaging Sciences*.

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[VL08] B. Vallet and B. Lévy. Spectral geometry processing with manifold harmonics. *Computer Graphics Forum*.